

# LIEB-THIRRING BOUND FOR SCHRÖDINGER OPERATORS WITH BERNSTEIN FUNCTIONS OF THE LAPLACIAN

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## Abstract

A Lieb-Thirring bound for Schrödinger operators with Bernstein functions of the Laplacian is shown by functional integration techniques. Several specific cases are discussed in detail.

*Keywords:* Bernstein functions, subordinate Brownian motion, heat kernel, non-local operators, fractional Laplacian, Schrödinger operator, Lieb-Thirring inequality

# 1 Introduction

In mathematical physics there is much interest in an inequality due originally to Lieb and Thirring giving an upper bound on the number of bound states for a Schrödinger operator  $-\frac{1}{2}\Delta + V$ . With  $N_0$  denoting the number of non-positive eigenvalues of the Schrödinger operator, in a semi-classical description it is expected that

$$N_0(V) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_{\{(p,x): |p|^2 + V(x) \leq 0\}} dp dx. \quad (1.1)$$

The right hand side above is computed as

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} \mathbb{1}_{\{|\xi| \leq \sqrt{V_-(x)}\}} d\xi = \frac{1}{(2\pi)^d} \frac{\sigma(S_{d-1})}{d} \int_{\mathbb{R}^d} |V_-(x)|^{d/2} dx \quad (1.2)$$

where  $\sigma(S_{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$  and  $V_-$  is the negative part of  $V$ . The Lieb-Thirring inequality then says that

$$N_0(V) \leq C_d \int_{\mathbb{R}^d} |V_-(x)|^{d/2} dx, \quad (1.3)$$

see [Lie76, Lie80], where  $C_d$  is a constant dependent on  $d$  alone. Various extensions have been further studied by many authors, see [LS10] and references therein.

Following our work [HIL09] in which we defined generalized Schrödinger operators of the form

$$H = \Psi \left( -\frac{1}{2}\Delta \right) + V \quad (1.4)$$

where  $\Psi$  denotes a Bernstein function (see below), it is a natural question if a similar Lieb-Thirring bound can be established and how does this depend on the choice of the Bernstein function. We will actually derive under some conditions that

$$N_0(V) \leq A \int_{\mathbb{R}^d} (\Psi^{-1}(|V(x)|))^{d/2} dx \quad (1.5)$$

(Theorem 3.23 and Corollary 3.9) by using estimates of the diagonal part of the heat kernel of subordinate Brownian motion generated by  $\Psi \left( -\frac{1}{2}\Delta \right)$ . This extension includes beside usual Schrödinger operators also fractional Schrödinger operators of the form  $(-\Delta)^{\alpha/2} + V$  and relativistic Schrödinger operators  $(-\Delta + m^2)^{1/2} - m + V$ . General Bernstein functions receive increasing attention in the study of stochastic processes with jump discontinuities and their potential theory [SSV10].

A Lieb-Thirring bound for generalized kinetic energy terms was first obtained in [Dau83]. Although the author mentions that similar bounds can be derived for generalizations using (1.4), the focus of that paper is primarily the relativistic Schrödinger operators above with or without mass. Lieb-Thirring inequalities for fractional Schrödinger operators compensated by the Hardy weight have been obtained more recently in

[FLS08] by using methods of Sobolev inequalities. A reference considering the same problem for relativistic Schrödinger operators including magnetic fields is [IMP07].

The remainder of this paper is organized as follows. In Section 2 we recall the definition of such Schrödinger operators and briefly describe the stochastic processes related to them. In the main Section 3 we state and prove the Lieb-Thirring inequality for this class of operators, and obtain some explicit variants. In Section 4 we discuss some cases of special interest.

## 2 Schrödinger operators with Bernstein functions of the Laplacian

Consider the function space

$$\mathcal{B} = \left\{ \Psi \in C^\infty(\mathbb{R}^+) : \Psi(x) \geq 0, (-1)^n \left( \frac{d^n \Psi}{dx^n} \right)(x) \leq 0, \forall n = 1, 2, \dots \right\}$$

An element of  $\mathcal{B}$  is called a *Bernstein function*. We also define the subclass  $\mathcal{B}_0 = \{f \in \mathcal{B} : \lim_{u \rightarrow 0+} f(u) = 0\}$ .

Bernstein functions in  $\mathcal{B}_0$  have the following integral representation. Let  $\mathcal{L}$  be the set of Borel measures  $\lambda$  on  $\mathbb{R} \setminus \{0\}$  such that  $\lambda((-\infty, 0)) = 0$  and  $\int_{\mathbb{R} \setminus \{0\}} (y \wedge 1) \lambda(dy) < \infty$ .

Note that every  $\lambda \in \mathcal{L}$  is a Lévy measure. Then it can be shown that for every Bernstein function  $\Psi \in \mathcal{B}_0$  there exists  $(b, \lambda) \in [0, \infty) \times \mathcal{L}$  such that

$$\Psi(u) = bu + \int_0^\infty (1 - e^{-uy}) \lambda(dy). \quad (2.1)$$

Conversely, the right hand side of (2.1) is in  $\mathcal{B}_0$  for each pair  $(b, \lambda) \in [0, \infty) \times \mathcal{L}$ . It is known that the map  $\mathcal{B}_0 \rightarrow [0, \infty) \times \mathcal{L}$ ,  $\Psi \mapsto (b, \lambda)$  is bijective.

Next consider a probability space  $(\Omega_\nu, \mathcal{F}_\nu, \nu)$  and a stochastic process  $(T_t)_{t \geq 0}$  on it. Recall that  $(T_t)_{t \geq 0}$  is called a subordinator whenever it is a Lévy process starting at 0, and  $t \mapsto T_t$  is almost surely a non-decreasing function. Let  $\mathcal{S}$  denote the set of subordinators on  $(\Omega_\nu, \mathcal{F}_\nu, \nu)$ . Also, let  $\Psi \in \mathcal{B}_0$  or, equivalently, a pair  $(b, \lambda) \in [0, \infty) \times \mathcal{L}$  be given. Then by the above bijection there is a unique  $(T_t)_{t \geq 0} \in \mathcal{S}$  such that

$$\mathbb{E}_\nu^0[e^{-uT_t}] = e^{-t\Psi(u)}. \quad (2.2)$$

Conversely, for every  $(T_t)_{t \geq 0} \in \mathcal{S}$  there exists a unique  $\Psi \in \mathcal{B}_0$ , i.e., a pair  $(b, \lambda) \in [0, \infty) \times \mathcal{L}$  such that (2.2) is satisfied. In particular, (2.1) coincides with the Lévy-Khintchine formula for Laplace exponents of subordinators. Using the bijection between  $\mathcal{B}_0$  and  $\mathcal{S}$ , we denote by  $T_t^\Psi$  the subordinator uniquely associated with  $\Psi \in \mathcal{B}_0$ .

It is known that the composition of a Brownian motion and a subordinator yields a Lévy process. This process is  $X_t : \Omega_P \times \Omega_\nu \ni (\omega_1, \omega_2) \mapsto B_{T_t(\omega_2)}(\omega_1) \in \mathbb{R}^d$  called

$d$ -dimensional subordinate Brownian motion with respect to the subordinator  $(T_t)_{t \geq 0}$ . Its properties are determined by  $\mathbb{E}_{P \times \nu}^0[e^{i\xi \cdot X_t}] = e^{-t\P(|\xi|^2/2)}$ . The function

$$P_t^\Psi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t\P(|\xi|^2/2)} d\xi \quad (2.3)$$

gives the distribution of  $X_t$  in  $\mathbb{R}^d$ .

Let  $h = -\Delta$  be the Laplacian in  $L^2(\mathbb{R}^d)$ . We assume throughout this paper that  $d \geq 3$ . Define the operator  $\Psi(h/2)$  on  $L^2(\mathbb{R}^d)$  with Bernstein function  $\Psi \in \mathcal{B}_0$ . Let  $V = V_+ - V_-$ , where  $V_+ = \max\{V, 0\}$ ,  $V_- = \min\{-V, 0\}$ , and assume that  $V_-$  is form-bounded with respect to  $\Psi(h/2)$  with a relative bound strictly smaller than 1, and  $V_+ \in L_{\text{loc}}^1(\mathbb{R}^d)$ . Then we define the *Schrödinger operator with Bernstein function  $\Psi$  of the Laplacian* by

$$H^\Psi = \Psi(h/2) \dot{+} V_+ \dot{-} V_-. \quad (2.4)$$

In what follows we simply write  $H^\Psi = \Psi(h/2) + V$  instead of (2.4).

**Proposition 2.1** *We have the functional integral representation for the semigroup  $e^{-tH^\Psi}$ ,  $t \geq 0$ , given by*

$$(f, e^{-tH^\Psi} g) = \int_{\mathbb{R}^d} dx \mathbb{E}_{P \times \nu}^x \left[ \overline{f(X_0)} g(X_t) e^{-\int_0^t V(X_s) ds} \right]. \quad (2.5)$$

PROOF. This is obtained by subordination and an application of the Trotter product formula combined with a limiting argument. For a detailed proof we refer to [HIL09, LHB11]. **qed**

In view of applications (quantum theory, anomalous transport theory, financial mathematics etc) some particular choices of Bernstein functions are of special interest involving the following stochastic processes:

- (1) *symmetric  $\alpha$ -stable processes*:  $\Psi(u) = (2u)^{\alpha/2}$ ,  $0 < \alpha \leq 2$
- (2) *relativistic  $\alpha$ -stable processes*:  $\Psi(u) = (2u + m^{2/\alpha})^{\alpha/2}$ , with  $m > 0$
- (3) *jump-diffusion processes*:  $\Psi(u) = au + bu^{\alpha/2}$ , with  $a, b \in \mathbb{R}$ .

### 3 Lieb-Thirring bound

The following is a standing assumption throughout the paper.

**Assumption 3.1**

- (1)  *$V$  is a continuous and non-positive function*
- (2) *there exists  $\lambda^* > 0$  such that  $\|(\Psi(h/2) + \lambda)^{-1/2} |V|^{1/2}\| < 1$  for all  $\lambda \geq \lambda^*$*
- (3) *the operator  $(\Psi(h/2) + \lambda)^{-1/2} |V|^{1/2}$  is compact for all  $\lambda \geq 0$*

(4) *there exists  $n_0 > 0$  such that  $\text{Tr}(|V|^{1/2}(\Psi(h/2) + \lambda)^{-1}|V|^{1/2})^n < \infty$  for all  $n \geq n_0$  and  $\lambda > 0$ .*

Part (2) of Assumption 3.1 implies that  $V$  is relatively form bounded with respect to  $\Psi(h/2)$  with relative bound strictly smaller than 1. Part (3) ensures that the Birman-Schwinger principle (3.3) holds, and (4) is used in the proof of Lemma 3.6.

**Example 3.2** Let  $L^{\infty,0}(\mathbb{R}^d)$  be the set of functions  $f \in L^\infty(\mathbb{R}^d)$  such that  $\lim_{|x| \rightarrow \infty} |f(x)| = 0$ . It is well known that if  $P, Q \in L^{\infty,0}(\mathbb{R}^d)$ , then  $P(-i\nabla)Q(x)$  is a compact operator [Sim04]. Thus  $(\Psi(h/2) + \lambda)^{-1/2}|V|^{1/2}$  is compact for  $V \in L^{\infty,0}$ , since  $\Psi$  is increasing. Moreover, if  $\Psi(h/2) = -\Delta$  and  $V \in L^{d/2}(\mathbb{R}^d)$ , (4) of Assumption 3.1 is satisfied with  $n_0 = d/2$ .

Consider the number

$$N_E(V) = \dim \mathbb{1}_{(-\infty, -E]}(H^\Psi). \quad (3.1)$$

In the original context of quantum theory this expression has the relevance of counting the number of bound states of energy up to  $-E < 0$ . Recall [Sim05] that the Birman-Schwinger kernel is defined by

$$K_E = |V|^{1/2}(\Psi(h/2) + E)^{-1}|V|^{1/2} \quad (3.2)$$

and the Birman-Schwinger principle says that

$$\begin{aligned} N_E(V) &= \dim \mathbb{1}_{[1, \infty)}(K_E), \quad -E < 0 \\ N_0(V) &\leq \dim \mathbb{1}_{[1, \infty)}(K_0), \quad E = 0. \end{aligned} \quad (3.3)$$

**Example 3.3** Let  $V = V_+ - V_-$  be such that  $V_- \in L^\infty(\mathbb{R}^d)$ . Since  $\Psi(h/2) - V_- \leq H^\Psi$ , the number of negative eigenvalues of  $H^\Psi$  is smaller than that of  $H_-^\Psi = \Psi(h/2) - V_-$ . So instead of  $H^\Psi$ , we consider  $H_-^\Psi$ . Since  $|V_-| \in L^\infty$ ,  $(\Psi(h/2) + \lambda)^{-1/2}|V_-|^{1/2}$  is compact. Thus the Birman-Schwinger principle can be applied to  $H_-^\Psi$ .

Let  $F_\lambda(x) = x(1 + \lambda x)^{-1} = x \int_0^\infty e^{-y(1+\lambda x)} dy$  and  $g_\lambda(x) = e^{-\lambda x}$ . The two functions are related by

$$F_\lambda(x) = x \int_0^\infty e^{-y} g_\lambda(xy) dy. \quad (3.4)$$

By a direct computation we obtain

$$F_\lambda(K_E) = |V|^{1/2}(\Psi(h/2) + \lambda|V| + E)^{-1}|V|^{1/2} \quad (3.5)$$

and by Laplace transform

$$(F_\lambda(K_E)u)(x) = |V(x)|^{1/2} \left( \int_0^\infty dt e^{-tE} e^{-t(\Psi(h/2) + \lambda|V|)} |V|^{1/2} u \right)(x) \quad (3.6)$$

follows. By (3.3) we have

$$\begin{aligned} N_E(V) &= \#\{F_\lambda(\mu) | F_\lambda(\mu) \text{ is an eigenvalue of } F_\lambda(K_E) \text{ and } \mu \geq 1\}, \quad E > 0 \\ N_0(V) &\leq \#\{F_\lambda(\mu) | F_\lambda(\mu) \text{ is an eigenvalue of } F_\lambda(K_0) \text{ and } \mu \geq 1\}, \quad E = 0. \end{aligned}$$

Since  $F_\lambda$  is monotone increasing, it follows that

$$N_E(V) \leq \frac{1}{F_\lambda(1)} \sum_{\substack{\mu \in \text{Spec}(K_E) \\ \mu \geq 1}} F_\lambda(\mu). \quad (3.7)$$

Using this we will estimate the trace of  $F_\lambda(K_E)$ . From Theorem 2.1 we obtain

$$(F_\lambda(K_E)u)(x) = |V(x)|^{1/2} \int_0^\infty dt e^{-tE} \mathbb{E}_{P \times \nu}^x \left[ e^{-\lambda \int_0^t |V(X_s)| ds} |V(X_t)|^{1/2} u(X_t) \right]. \quad (3.8)$$

In order to express the kernel of  $e^{-t(\Psi(h/2)+\lambda|V|)}$  in terms of a conditional expectation we use the following notation. Let  $\mathbb{E}_{P \times \nu}^0[Y|X_t]$  be conditional expectation with respect to the  $\sigma$ -field  $\sigma(X_t)$ , i.e.,  $\mathbb{E}_{P \times \nu}^0[Y|X_t]$  is measurable with respect to  $\sigma(X_t)$ . Generally, a function  $f$  measurable with respect to  $\sigma(X_t)$  can be written as  $f = g(X_t)$  with a suitable function  $g$ . We write  $\mathbb{E}_{P \times \nu}^0[Y|X_t] = g(X_t)$ , and use the notation  $g(x) = \mathbb{E}_{P \times \nu}^0[Y|X_t = x]$ , i.e.,  $\mathbb{E}_{P \times \nu}^0[Y|X_t] = \int \mathbb{E}_{P \times \nu}^0[Y|X_t = x] P_t^\Psi(x) dx$ . In these terms we then have

$$e^{-t(\Psi(h/2)+\lambda|V|)}(x, y) = \mathbb{E}_{P \times \nu}^0 \left[ e^{-\lambda \int_0^t |V(X_s+x)| ds} \Big| X_t + x = y \right] P_t^\Psi(x - y), \quad (3.9)$$

where  $P_t^\Psi$  is the distribution of  $X_t$  given by (2.3).

**Lemma 3.4** *The map  $(x, y) \mapsto e^{-t(\Psi(h/2)+\lambda|V|)}(x, y)$  is continuous.*

PROOF. Let  $P_{[0,T]}^{x,y}$  denote Brownian bridge measure starting from  $x$  at  $t = 0$  and ending in  $y$  at  $t = T$ . Then by the Feynman-Kac-like formula (2.5) and using that  $X_s = B_{T_s}$  we see that

$$(f, e^{-t(\Psi(h/2)+\lambda|V|)}g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \bar{f}(x) g(y) \mathbb{E}_\nu \left[ \Pi_{T_t}(x - y) \mathbb{E}_{P_{[0,T_t]}^{x,y}} \left[ e^{-\int_0^t |V(B_{T_s})| ds} \right] \right] dx dy, \quad (3.10)$$

where  $\Pi_t(x)$  is the Gaussian heat kernel. Note that the measure  $P_{[0,T_t]}^{x,y} = P_{[0,T_t(\omega_2)]}^{x,y}$  is defined for every  $\omega_2 \in \Omega_\nu$ . For every  $\omega_2 \in \Omega_\nu$  we also define the Brownian bridge  $(Z_t)_{t \geq 0}$  by

$$Z_t = \left(1 - \frac{t}{T_t}\right) x + \frac{t}{T_t} y - \frac{t}{T_t} B_{T_t} + B_t,$$

where  $T_t$  depends on  $\omega_2$ . Thus (3.10) is equal to

$$(f, e^{-t(\Psi(h/2)+\lambda|V|)}g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \bar{f}(x) g(y) \mathbb{E}_\nu \left[ \Pi_{T_t}(x - y) \mathbb{E}_P^0 \left[ e^{-\int_0^t |V(Z_s)| ds} \right] \right] dx dy. \quad (3.11)$$

Hence the integral kernel is given by

$$e^{-t(\Psi(h/2)+|V|)}(x, y) = \mathbb{E}_\nu \left[ \Pi_{T_t}(x - y) \mathbb{E}_P^0 [e^{-\int_0^t |V(Z_s)| ds}] \right]$$

and implies joint continuity with respect to  $(x, y)$ . **qed**

From Lemma 3.4 it follows that the kernel of  $F_\lambda(K_E)$ ,

$$\begin{aligned} F_\lambda(K_E)(x, y) &= |V(x)|^{1/2} |V(y)|^{1/2} \\ &\times \int_0^\infty dt e^{-tE} \mathbb{E}_{P \times \nu}^0 \left[ g_\lambda \left( \int_0^t |V(X_s + x)| ds \right) \middle| X_t + x = y \right] P_t^\Psi(x - y) \end{aligned} \quad (3.12)$$

is also jointly continuous in  $(x, y)$ . Here we used that  $g(x) = e^{-\lambda x}$ . By setting  $x = y$  in (3.12) it is seen that  $\text{Tr } F_\lambda(K_E) = \int_{\mathbb{R}^d} F_\lambda(K_E)(x, x) dx$ . This gives the expression

$$\text{Tr } F_\lambda(K_E) = \int_{\mathbb{R}^d} dx |V(x)| \int_0^\infty dt e^{-tE} \mathbb{E}_{P \times \nu}^0 \left[ g_\lambda \left( \int_0^t |V(X_s + x)| ds \right) \middle| X_t = 0 \right] P_t^\Psi(0). \quad (3.13)$$

**Lemma 3.5** *It follows that*

$$\text{Tr } F_\lambda(K_E) = \int_{\mathbb{R}^d} dx \int_0^\infty \frac{dt}{t} e^{-tE} \mathbb{E}_{P \times \nu}^0 \left[ G_\lambda \left( \int_0^t |V(X_s + x)| ds \right) \middle| X_t = 0 \right] P_t^\Psi(0), \quad (3.14)$$

where  $G_\lambda(x) = x g_\lambda(x) = x e^{-\lambda x}$ .

PROOF. It suffices to show that

$$\begin{aligned} \frac{1}{t} \int_{\mathbb{R}^d} dx \mathbb{E}_{P \times \nu}^0 &\left[ e^{-\int_0^t |V(X_s + x)| ds} \int_0^t |V(X_r + x)| dr \middle| X_t = 0 \right] P_t^\Psi(0) \\ &= \int_{\mathbb{R}^d} dx |V(x)| \mathbb{E}_{P \times \nu}^0 \left[ e^{-\int_0^t |V(X_s + x)| ds} \middle| X_t = 0 \right] P_t^\Psi(0). \end{aligned} \quad (3.15)$$

Let  $U_r = e^{-r(\Psi(h/2)+|V|)}|V|e^{-(t-r)(\Psi(h/2)+|V|)}$  for  $0 \leq r \leq t$ . Note that  $U_r$  is compact and thus  $\text{Tr } U_r = \text{Tr } U_0$ . By the Markov property of  $(X_t)_{t \geq 0}$  it follows that

$$\begin{aligned} (U_r f)(x) &= \mathbb{E}_{P \times \nu}^x \left[ e^{-\int_0^r |V(X_s)| ds} |V(X_r)| \mathbb{E}_{P \times \nu}^{X_r} \left[ e^{-\int_0^{t-r} |V(X_s)| ds} f(X_{t-r}) \right] \right] \\ &= \mathbb{E}_{P \times \nu}^x \left[ e^{-\int_0^t |V(X_s)| ds} |V(X_r)| f(X_t) \right]. \end{aligned}$$

Thus the right hand side above is expressed as

$$= \int_{\mathbb{R}^d} P_t^\Psi(x - y) \mathbb{E}_{P \times \nu}^0 \left[ e^{-\int_0^t |V(X_s + x)| ds} |V(X_r + x)| \middle| X_t + x = y \right] f(y) dy.$$

This furthermore gives

$$\mathrm{Tr} U_r = \int_0^t \frac{dr}{t} \mathrm{Tr} U_r = \frac{1}{t} \int_{\mathbb{R}^d} dx P_t^\Psi(0) \mathbb{E}_{P \times \nu}^0 \left[ e^{-\int_0^t |V(X_s+x)| ds} \int_0^t |V(X_r+x)| dr \middle| X_t = 0 \right], \quad (3.16)$$

where we interchanged  $dr$  and  $dP^0$ . Equality  $\int_0^t \frac{dr}{t} \mathrm{Tr} U_r = \mathrm{Tr} U_0$  together with (3.16) yield (3.15). Hence the lemma follows. **qed**

We may vary  $F_\lambda$  and  $g_\lambda$  while keeping relationship (3.4) unchanged. Let  $F : [0, \infty) \rightarrow [0, \infty)$  be a strictly increasing function such that

$$F(x) = x \int_0^\infty e^{-y} g(xy) dy, \quad (3.17)$$

where  $g$  is a non-negative function on  $\mathbb{R}$ . Write

$$G(x) = xg(x). \quad (3.18)$$

**Lemma 3.6** *Let Assumption 3.1 hold and take any  $F$ ,  $G$  and  $g$  satisfying (3.17). Suppose that  $G$  is non-negative and lower semi-continuous. Then it follows that*

$$\mathrm{Tr} F(K_E) = \int_{\mathbb{R}^d} dx \int_0^\infty dt \frac{e^{-tE}}{t} \mathbb{E}_{P \times \nu}^0 \left[ G \left( \int_0^t |V(X_s+x)| ds \right) \middle| X_t = 0 \right] P_t^\Psi(0). \quad (3.19)$$

PROOF. The proof is obtained by a slight modification of [Sim04, Theorem 8.2] and [LHB11, Lemma 3.51]. **qed**

**Theorem 3.7 (Lieb-Thirring bound)** *Let Assumption 3.1 hold,  $F$ ,  $G$  be any functions satisfying (3.17) and (3.18), and  $G$  furthermore be convex. Then*

$$N_0(V) \leq \frac{1}{F(1)} \int_0^\infty \frac{ds}{s} G(s) \int_{\mathbb{R}^d} P_{s/|V(x)|}^\Psi(0) \mathbb{1}_{\{|V(x)|>0\}} dx, \quad (3.20)$$

where

$$P_{s/|V(x)|}^\Psi(0) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-s\Psi(|\xi|^2/2)/|V(x)|} d\xi.$$

We note that the right hand side of (3.20) may not be finite, this depends on the choice of the convex function  $G$ .

PROOF. Since  $F$  is a monotone increasing function, we have

$$\begin{aligned} N_0(V) &\leq \frac{1}{F(1)} \mathrm{Tr}(F(K_0)) \\ &= \frac{1}{F(1)} \int_0^\infty \frac{dt}{t} \int_{\mathbb{R}^d} dx \mathbb{E}_{P \times \nu}^0 \left[ G \left( \int_0^t |V(X_s+x)| \frac{ds}{t} \right) \middle| X_t = 0 \right] P_t^\Psi(0). \end{aligned}$$



Then by the Jensen inequality

$$N_0(V) \leq \frac{1}{F(1)} \int_0^\infty \frac{dt}{t} \int_{\mathbb{R}^d} dx \mathbb{E}_{P \times \nu}^0 \left[ \int_0^t \frac{ds}{t} G(t|V(X_s + x)|) \middle| X_t = 0 \right] P_t^\Psi(0).$$

Using that  $\int_0^t \frac{ds}{t} = 1$  and swapping  $dx$  and  $dP^0 \times d\nu$ , we obtain

$$N_0(V) \leq \frac{1}{F(1)} \int_0^\infty P_t^\Psi(0) \frac{dt}{t} \int_{\mathbb{R}^d} G(t|V(x)|) dx.$$

When  $V(x) = 0$ , also  $G(tV(x)) = 0$ . This implies that the right hand side above equals

$$\frac{1}{F(1)} \int_0^\infty P_t^\Psi(0) \frac{dt}{t} \int_{\mathbb{R}^d} G(t|V(x)|) \mathbb{1}_{\{|V(x)| > 0\}} dx.$$

Changing the variable from  $t|V(x)|$  to  $s$  and integrating with respect to  $s$ , we obtain (3.20). **qed**

Next we are interested to see how the Lieb-Thirring bound (3.20) in fact depends on the Bernstein function  $\Psi$ . To make this expression more explicit we note that the diagonal part of the heat kernel has the representation [JKLS12]

$$P_t^\Psi(0) = (2\pi)^{-d} \int_0^\infty e^{-r} \left( \int_{\mathbb{R}^d} \mathbb{1}_{\{\sqrt{\Psi(\xi^2/2)} \leq \sqrt{r/t}\}} d\xi \right) dr. \quad (3.21)$$

Denote by  $\mathbb{B}^\Psi(x, r)$  a ball of radius  $r$  centered in  $x$  in the topology of the metric

$$d^\Psi(\xi, \eta) = \sqrt{\Psi(|\eta - \xi|^2/2)}.$$

Notice that  $d^\Psi(\xi, \eta) = 0$  if and only if  $\xi = \eta$ , since  $\Psi$  is concave and a  $C^\infty$ -function. Then the integral  $\int_{\mathbb{R}^d} \mathbb{1}_{\{\sqrt{\Psi(\xi^2/2)} \leq \sqrt{r/t}\}} d\xi$  is the volume of  $\mathbb{B}^\Psi(0, \sqrt{r/t})$  in this metric. If  $d^\Psi$  satisfies the condition

$$\int_{\mathbb{R}^d} \mathbb{1}_{\mathbb{B}^\Psi(x, 2r)} dy \leq c \int_{\mathbb{R}^d} \mathbb{1}_{\mathbb{B}^\Psi(x, r)} dy, \quad x \in \mathbb{R}^d, r > 0$$

with a constant  $c > 0$  independent of  $x$  and  $r$ , then  $d^\Psi$  is said to have the volume doubling property. When  $d^\Psi$  has this property, then furthermore it follows that

$$c_1 \int_{\mathbb{R}^d} \mathbb{1}_{\{\sqrt{\Psi(\xi^2/2)} \leq \sqrt{r/t}\}} d\xi \leq P_t^\Psi(0) \leq c_2 \int_{\mathbb{R}^d} \mathbb{1}_{\{\sqrt{\Psi(\xi^2/2)} \leq \sqrt{r/t}\}} d\xi \quad (3.22)$$

with some constants  $c_1$  and  $c_2$ . A necessary and sufficient condition for  $\Psi \in \mathcal{B}_0$  to give rise to a volume doubling  $d^\Psi$  is

$$\liminf_{u \rightarrow 0} \frac{\Psi(Cu)}{\Psi(u)} > 1 \quad \text{and} \quad \liminf_{u \rightarrow \infty} \frac{\Psi(Cu)}{\Psi(u)} > 1$$

for some  $C > 1$ . In particular, this implies that  $\Psi$  increases at infinity as a (possibly fractional) power. For details, we refer to [JKLS12].

**Theorem 3.8** *Suppose that  $\Psi \in \mathcal{B}_0$  is strictly monotone increasing. Then under the assumptions of Theorem 3.7 we have*

$$N_0(V) \leq \frac{2^{\frac{3d}{2}+1}\pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})F(1)} \int_0^\infty \frac{ds}{s} G(s) \int_{\mathbb{R}^d} dx \int_0^\infty \left( \Psi^{-1} \left( \frac{r|V(x)|}{s} \right) \right)^{d/2} e^{-r} dr. \quad (3.23)$$

Furthermore, if  $d^\Psi$  has the volume doubling property, then

$$N_0(V) \leq c_2 \frac{2^{\frac{3d}{2}+1}\pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})F(1)} \int_0^\infty \frac{ds}{s} G(s) \int_{\mathbb{R}^d} \left( \Psi^{-1} \left( \frac{|V(x)|}{s} \right) \right)^{d/2} dx. \quad (3.24)$$

PROOF. Since under the assumption the function  $\Psi \in \mathcal{B}_0$  is invertible and its inverse is increasing, the proof is straightforward using  $\text{Ker } \Psi = \{0\}$ , (3.21) and (3.22). **qed**

In the case when  $\Psi \in \mathcal{B}_0$  has a scaling property, we can derive a more explicit formula.

**Corollary 3.9** *Suppose that  $\Psi \in \mathcal{B}_0$  is strictly monotone increasing and the assumptions of Theorem 3.7 hold. In addition, assume that there exists  $\gamma > 0$  such that  $\Psi(au) = a^\gamma \Psi(u)$  for all  $a, u \geq 0$ . Then*

$$N_0(V) \leq A \int_{\mathbb{R}^d} (\Psi^{-1}(|V(x)|))^{d/2} dx, \quad (3.25)$$

$$\text{where } A = \frac{2^{\frac{3d}{2}+1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2\gamma} + 1)}{d\Gamma(\frac{d}{2})F(1)} \int_0^\infty G(s)s^{-1-\frac{d}{2\gamma}} ds.$$

PROOF. The inverse function  $\Psi^{-1}$  has the scaling property  $\Psi^{-1}(av) = a^{1/\gamma}\Psi^{-1}(v)$ . Thus the corollary follows. **qed**

Instead of the scaling property suppose now that there exists  $\lambda > 0$  such that  $\Psi(u) \geq Cu^\lambda$  with a constant  $C > 0$ . This inequality holds for at least large enough  $u$  if  $d^\Psi$  has the volume doubling property. Then we have a similar formula to that in Corollary 3.9.

**Corollary 3.10** *Suppose that  $\Psi \in \mathcal{B}_0$  is strictly monotone increasing and the assumptions of Theorem 3.7 hold. If  $\Psi(u) \geq Cu^\lambda$ , then*

$$N_0(V) \leq A \int_{\mathbb{R}^d} |V(x)|^{d/2\lambda} dx, \quad (3.26)$$

$$\text{where } A = \frac{2^{\frac{3d}{2}+1}\pi^{\frac{d}{2}}C^{-1/\lambda}}{d\Gamma(\frac{d}{2})F(1)} \int_0^\infty G(s)s^{-1-\frac{d}{2\lambda}} ds.$$

PROOF.  $\Psi(u) \geq Cu^\lambda$  gives  $\Psi^{-1}(u) \leq C^{-1/\lambda}u^{1/\lambda}$ . Then the corollary follows. **qed**

In some special cases of Bernstein functions  $\Psi$  we can derive more explicit forms of the Lieb-Thirring inequality.

## 4 Specific cases

### 4.1 Fractional Schrödinger operators (symmetric $\alpha$ -stable processes)

Let  $\Psi(u) = (2u)^{\alpha/2}$  and  $H^\Psi = (-\Delta)^{\alpha/2}$ . Throughout this section we suppose that  $0 < \alpha \leq 2$ . Define the quadratic form

$$Q(f, g) = ((-\Delta)^{\alpha/4} f, (-\Delta)^{\alpha/4} g) - (|V|^{1/2} f, |V|^{1/2} g). \quad (4.1)$$

Boundedness from below of the cases  $\alpha = 1$  and  $\alpha = 2$  is proven in [LL01].

**Lemma 4.1** *Let  $V \in L^{d/\alpha}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ . Then  $V$  is form bounded with respect to  $(-\Delta)^{\alpha/2}$  with a relative bound strictly smaller than 1. In particular, we have that*

$$\inf_{f \in D((-\Delta)^{\alpha/4})} Q(f, f) > -\infty.$$

**PROOF.** Let  $I_\alpha = (-\Delta)^{-\alpha/2}$  be the operator of the Riesz potential. Recall the Sobolev inequality  $\|I_\alpha f\|_q \leq C\|f\|_p$  for  $q = \frac{pd}{d-\alpha p}$  and  $d > \alpha p$ . From this we obtain

$$\|f\|_q \leq C\|(-\Delta)^{\alpha/2} f\|_p \quad (4.2)$$

with some constant  $C$ . Hence it follows that

$$\|(-\Delta)^{\alpha/4} f\|_2^2 \geq \frac{1}{C} \|f\|_{\frac{2d}{d-2\alpha}}^2 \geq \frac{1}{C} (|V|^{1/2} f, |V|^{1/2} f) \|V\|_{d/\alpha}^{-1}. \quad (4.3)$$

The estimate gives  $Q(f, f) \geq 0$  when  $\|V\|_{d/\alpha} < 1/C$ . Let  $V(x) = v(x) + w(x)$  be such that  $v \in L^{d/\alpha}(\mathbb{R}^d)$  and  $w \in L^\infty(\mathbb{R}^d)$ . Then there is a bounded function  $\lambda(x)$  such that  $h = v - \lambda$  satisfies that  $\|h\|_{d/\alpha} < 1/C$ . Thus  $V = h + (w + \lambda)$  and  $w + \lambda \in L^\infty(\mathbb{R}^d)$ , and the lemma follows. **qed**

**Corollary 4.2** *Let  $\Psi(u) = (2u)^{\alpha/2}$  and let Assumption 3.1 hold. If  $V \in L^{d/\alpha}(\mathbb{R}^d)$ , then there exists a constant  $\mathbf{L}_{\alpha,d}$  independent of  $V$  such that*

$$N_0(V) \leq \mathbf{L}_{\alpha,d} \int_{\mathbb{R}^d} |V(x)|^{d/\alpha} dx, \quad 0 < \alpha \leq 2. \quad (4.4)$$

**PROOF.** We have that

$$P_t^\Psi(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} d\xi = \frac{C(\alpha, d)}{t^{d/\alpha}}, \quad (4.5)$$

where  $C(\alpha, d) = \frac{\sigma(S_{d-1})\Gamma(d/\alpha)}{\alpha(2\pi)^d}$ . Thus the corollary follows from Theorem 3.7 with the constant prefactor

$$\mathbf{L}_{\alpha,d} = \frac{C(\alpha, d)}{F(1)} \int_0^\infty s^{-1-d/\alpha} G(s) ds.$$

**qed**

This proof was obtained by hand through direct heat kernel estimates, however, the result also follows by either of Corollaries 3.9 or 3.10.

## 4.2 Relativistic Schrödinger operators (relativistic Cauchy processes)

Let  $\Psi(u) = \sqrt{2u + m^2} - m$  and  $H^\Psi = (-\Delta + m^2)^{1/2} - m$ . By using (4.3) we derive that

$$\|(-\Delta + m^2)^{1/4} f\|_2^2 \geq \|(-\Delta)^{1/4} f\|_2^2 \geq \frac{1}{C} \|f\|_{\frac{2d}{d-2}}^2 \geq \frac{1}{C} (|V|^{1/2} f, |V|^{1/2} f) \|V\|_d^{-1}. \quad (4.6)$$

Hence  $V \in L^{d/2}(\mathbb{R}^d)$  is relatively form bounded with respect to  $(-\Delta + m^2)^{1/2} - m$  with relative bound strictly smaller than 1.

**Corollary 4.3** *Let  $\Psi(u) = \sqrt{2u + m^2} - m$ . Let Assumption 3.1 hold, and suppose that  $V \in L^d(\mathbb{R}^d)$  if  $m = 0$ , and  $V \in L^{d/2}(\mathbb{R}^d) \cap L^d(\mathbb{R}^d)$  if  $m \neq 0$ . Then there exist  $\mathbf{L}_{1,d}^{(1)}$ ,  $\mathbf{L}_{1,d}^{(2)}$  and  $\mathbf{L}_{1,d}^{(3)}$  independent of  $V$  such that*

$$N_0(V) \leq \mathbf{L}_{1,d}^{(1)} \int_{\mathbb{R}^d} |V(x)|^d dx \quad m = 0 \quad (4.7)$$

$$N_0(V) \leq \mathbf{L}_{1,d}^{(2)} \int_{\mathbb{R}^d} |V(x)|^d dx + \mathbf{L}_{1,d}^{(3)} \int_{\mathbb{R}^d} |V(x)|^{d/2} dx \quad m \neq 0.$$

PROOF. The proof for  $m = 0$  can be reduced to Corollary 4.2 with  $\alpha = 1$ . Let  $m > 0$ . We have

$$P_t^\Psi(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t(\sqrt{|\xi|^2 + m^2} - m)} d\xi.$$

A computation (see Corollary 4.4 below) gives

$$P_t^\Psi(0) \leq \frac{C_1(d)}{t^d} + \frac{C_2(d)}{t^{d/2}} \quad (4.8)$$

with some positive constants  $C_1(d)$ , and  $C_2(d)$ . Hence we have

$$N_0(V) \leq \frac{1}{F(1)} \left( C_1(d) \int_{\mathbb{R}^d} dx \int_0^\infty \frac{ds}{s^{1+d}} G(s) |V(x)|^d + C_2(d) \int_{\mathbb{R}^d} dx \int_0^\infty \frac{ds}{s^{1+d/2}} G(s) |V(x)|^{d/2} \right)$$

for  $m \neq 0$ . Thus the corollary follows with

$$\begin{aligned} \mathbf{L}_{1,d}^{(1)} &= \frac{2(d-1)!}{(4\pi)^{d/2} \Gamma(d/2)} \frac{1}{F(1)} \int_0^\infty s^{-1-d} G(s) ds \\ \mathbf{L}_{1,d}^{(2)} &= \frac{2^{3d/2} (d-1)!}{F(1)} \int_0^\infty s^{-1-d} G(s) ds \\ \mathbf{L}_{1,d}^{(3)} &= \frac{2^{-1+3d/4} m^{d/2} \Gamma(d/2)}{F(1)} \int_0^\infty s^{-1-\frac{d}{2}} G(s) ds. \end{aligned}$$

qed

### 4.3 Fractional relativistic Schrödinger operators (relativistic $\alpha$ -stable processes)

Let  $\Psi(u) = (2u + m^{2/\alpha})^{\alpha/2} - m$  and  $H^\Psi = (-\Delta + m^{2/\alpha})^{\alpha/2} - m$ . Using (4.3) we can also derive that

$$\|(-\Delta + m^{2/\alpha})^{\alpha/4} f\|_2^2 \geq \|(-\Delta)^{\alpha/4} f\|_2^2 \geq \frac{1}{C} \|f\|_{\frac{2d}{d-2\alpha}}^2 \geq \frac{1}{C} (|V|^{1/2} f, |V|^{1/2} f) \|V\|_{d/\alpha}^{-1}. \quad (4.9)$$

Hence  $V \in L^{d/\alpha}(\mathbb{R}^d)$  is relatively form bounded with respect to  $(-\Delta + m^{2/\alpha})^{\alpha/2} - m$  with relative bound strictly smaller than 1.

**Corollary 4.4** *Let  $\Psi(u) = (2u + m^{2/\alpha})^{\alpha/2} - m$ ,  $\alpha \neq 1, 2$ . Let Assumption 3.1 hold, and suppose that  $V \in L^{d/\alpha}(\mathbb{R}^d)$  if  $m = 0$ , and  $V \in L^{d/\alpha}(\mathbb{R}^d) \cap L^{d/2}(\mathbb{R}^d)$  if  $m \neq 0$ . Then there exist  $\mathbf{L}_{\alpha,d}^{(1)}$ ,  $\mathbf{L}_{\alpha,d}^{(2)}$  and  $\mathbf{L}_{\alpha,d}^{(3)}$ , independent of  $V$  such that*

$$N_0(V) \leq \mathbf{L}_{\alpha,d}^{(1)} \int_{\mathbb{R}^d} |V(x)|^{d/\alpha} dx \quad m = 0 \quad (4.10)$$

$$N_0(V) \leq \mathbf{L}_{\alpha,d}^{(2)} \int_{\mathbb{R}^d} |V(x)|^{d/\alpha} dx + \mathbf{L}_{\alpha,d}^{(3)} \int_{\mathbb{R}^d} |V(x)|^{d/2} dx \quad m \neq 0.$$

PROOF. For  $m = 0$ , we adopt the proof of Corollary 4.2. Let  $m > 0$ , then

$$P_t^\Psi(0) = \frac{\sigma(S_{d-1})}{(2\pi)^d} \int_0^\infty e^{-t((r^2 + m^{2/\alpha})^{\alpha/2} - m)} r^{d-1} dr. \quad (4.11)$$

Using the inequality  $u^{\alpha/2} - 1 \leq \frac{\alpha}{2}(u - 1)$ ,  $0 \leq u \leq 1$ , for  $\alpha \in (0, 2)$ , and the substitution  $u = m^{2/\alpha}/(r^2 + m^{2/\alpha})$  it follows that

$$(r^2 + m^{2/\alpha})^{\alpha/2} - m \geq \frac{\alpha}{2} r^2 (r^2 + m^{2/\alpha})^{(\alpha/2)-1}. \quad (4.12)$$

Assuming that  $r \leq m^{1/\alpha}$ , i.e.,  $r^2 + m^{2/\alpha} \leq 2m^{2/\alpha}$ , it follows from (4.12) that

$$(r^2 + m^{2/\alpha})^{\alpha/2} - m \geq \frac{\alpha}{2} \frac{r^2}{(2m^{2/\alpha})^{1-\alpha/2}}. \quad (4.13)$$

If  $r > m^{1/\alpha}$ , i.e.,  $2r^2 > r^2 + m^{2/\alpha}$ , then it follows that

$$(r^2 + m^{2/\alpha})^{\alpha/2} - m \geq \frac{\alpha}{2^{2-\alpha/2}} r^\alpha. \quad (4.14)$$

Therefore, using (4.13) and (4.14) in (4.11), write

$$\int_0^\infty e^{-t((r^2 + m^{2/\alpha})^{\alpha/2} - m)} r^{d-1} dr \leq \int_{r \leq m^{1/\alpha}} e^{-\frac{\alpha r^2}{2(2m^{2/\alpha})^{1-\alpha/2}} t} r^{d-1} dr + \int_{r > m^{1/\alpha}} e^{-\frac{\alpha r^\alpha}{2^{2-\alpha/2}} t} r^{d-1} dr.$$

For the first integral, set  $u = \frac{\alpha r^2}{2(2m^{2/\alpha})^{1-\alpha/2}}t$  to obtain

$$\int_{r \leq m^{1/\alpha}} e^{-\frac{\alpha r^2}{2(2m^{2/\alpha})^{1-\alpha/2}}t} r^{d-1} dr \leq \frac{K_1^{d/2}}{2t^{d/2}} \int_0^\infty e^{-u} u^{(d/2)-1} du = \frac{C_2(\alpha, d)}{t^{d/2}}, \quad (4.15)$$

where  $C_2(\alpha, d) = \frac{K_1^{d/2} \Gamma(d/2)}{2}$ , and  $K_1 = \frac{2}{\alpha} (2m^{2/\alpha})^{1-\alpha/2}$ . For the second integral similarly we obtain that

$$\int_{r > m^{1/\alpha}} e^{-\frac{\alpha r^\alpha}{2^{2-\alpha/2}}t} r^{d-1} dr \leq \frac{1}{\alpha} \frac{K_2^d}{t^{d/\alpha}} \int_0^\infty e^{-u} u^{(d/\alpha)-1} du = \frac{C_3(\alpha, d)}{t^{d/\alpha}}, \quad (4.16)$$

where  $C_3(\alpha, d) = \frac{K_2^d \Gamma(d/\alpha)}{\alpha}$ , and  $K_2 = \left(\frac{2^{2-\alpha/2}}{\alpha}\right)^{1/\alpha}$ . Thus, using the results of (3.20) and (4.11) together with (4.15) and (4.16), we find the positive constants

$$\begin{aligned} \mathbf{L}_{\alpha, d}^{(2)} &= \frac{C_2(\alpha, d)}{F(1)} \int_0^\infty s^{-1-d/2} G(s) ds, \\ \mathbf{L}_{\alpha, d}^{(3)} &= \frac{C_3(\alpha, d)}{F(1)} \int_0^\infty s^{-1-d/\alpha} G(s) ds \end{aligned}$$

such that (4.10) holds for  $m \neq 0$ . Thus the corollary follows. qed

#### 4.4 Sums of different stable generators

Let  $\Psi(u) = (2u)^{\alpha/2} + (2u)^{\beta/2}$ ,  $0 < \alpha, \beta < 2$ ,  $\alpha \neq \beta$ , and  $H^\Psi = (-\Delta)^{\alpha/2} + (-\Delta)^{\beta/2} + V$ , acting in  $L^2(\mathbb{R}^d)$ . Relative boundedness of  $V$  follows similarly as in Lemma 4.1, whenever  $V \in L^{d/\alpha}(\mathbb{R}^d) \cap L^{d/\beta}(\mathbb{R}^d)$ . This is an example in which Corollary 3.9 does not apply, however, we have the following result.

**Corollary 4.5** *Suppose that Assumption 3.1 holds and  $V \in L^{d/\alpha}(\mathbb{R}^d) \cap L^{d/\beta}(\mathbb{R}^d)$ . Then*

$$N_0(V) \leq \mathbf{L}_\alpha \int_{\mathbb{R}^d} |V(x)|^{d/\alpha} dx + \mathbf{L}_\beta \int_{\mathbb{R}^d} |V(x)|^{d/\beta} dx, \quad (4.17)$$

where

$$\mathbf{L}_\alpha = \frac{c}{F(1)} \int_0^\infty s^{-1-d/\alpha} G(s) ds, \quad \mathbf{L}_\beta = \frac{c}{F(1)} \int_0^\infty s^{-1-d/\beta} G(s) ds.$$

PROOF. It is known [CK08] that

$$P_t^\Psi(0) \leq c \left( t^{-\frac{d}{\alpha}} \wedge t^{-\frac{d}{\beta}} \right), \quad t > 0 \quad (4.18)$$

with some constant  $c > 0$ . Then by (3.20) we obtain the claim. qed

## 4.5 Jump-diffusion operators

Let  $\Psi(u) = u + bu^{\alpha/2}$ ,  $\alpha \in (0, 2)$ , and  $b \in (0, 1]$ . Then we have  $H^\Psi = -\Delta + b(-\Delta)^{\alpha/2} + V$ . By (4.3) we see that when  $V \in L^{d/\alpha}(\mathbb{R}^d) \cup L^{d/2}(\mathbb{R}^d)$ ,  $V$  is relatively form bounded with respect to  $-\Delta + b(-\Delta)^{\alpha/2}$  with relative bound strictly smaller than 1.

**Corollary 4.6** *If Assumption 3.1 holds and  $V \in L^{\frac{d}{2} + \frac{d}{\alpha}}(\mathbb{R}^d)$ , then*

$$N_0(V) \leq \mathbf{L} \int_{\mathbb{R}^d} |V(x)|^{d/2} dx + \mathbf{L}_\alpha \int_{\mathbb{R}^d} |V(x)|^{d/\alpha} dx, \quad (4.19)$$

where

$$\mathbf{L} = \frac{c}{F(1)} \int_0^\infty s^{-1-d/2} G(s) ds, \quad \mathbf{L}_\alpha = \frac{c}{F(1)} \int_0^\infty s^{-1-d/\alpha} G(s) ds.$$

PROOF. In this case it is known [CKS11] that with some  $c > 0$

$$p_t^b(x - y) \leq (t^{-d/2} \wedge (bt)^{-d/\alpha}) \wedge \left( t^{-d/2} e^{-|x-y|^2/ct} + (bt)^{-d/\alpha} \wedge \frac{bt}{|x-y|^{d+\alpha}} \right),$$

and in the same way as in the previous examples the result follows. **qed**

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